Observation and Identification Via High-Order Sliding Modes

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1 Introduction

Observation of system states in the presence of unknown inputs is one of the most important problems in the modern control theory. Usually the observers for such systems are designed under assumption that only the outputs are available but not their derivatives. In particular, it is required that the unknown inputs need to match to the known outputs.

Sliding-mode-based robust state observation is successfully developed in the Variable Structure Theory within the recent years (see [1], [2], [3], [4], [5], [6], [7]). The sliding-mode-based observation has such attractive features as

- insensitivity (more than robustness) with respect to unknown inputs;
- possibility to use the equivalent output injection in order to obtain additional information (e.g., the reconstruction of the unknown inputs).

Further analysis has shown that these observers are very useful for fault detection [8], [9], [10]. However in those observers the fault detection is realized via equivalent output injection, while the estimation of the observable states were made by traditional smooth (usually Luenberger) observers without differentiators. It generates their main limitation: the output of the system should have a relative degree one with respect to the unknown input. This condition is very restrictive even for velocity observers for mechanical systems [11], [12], [13], [14], [15].

Step-by-step vector-state reconstruction by means of sliding modes is studied by [16], [17], [18]. These observers are based on a system transformation to a triangular form and successive estimation of the state vector using the equivalent output injection. Some sufficient conditions for observation of linear time-invariant systems with unknown inputs were obtained in [18]. Moreover the above-mentioned observers *theoretically* ensure finite-time convergence for all system states.

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Unfortunately, the realization of step-by-step sliding-mode observers is based on conventional sliding modes requiring filtration at each step due to imperfections of analog devices or discretization effects.

In order to avoid the filtration, the hierarchical observers were recently developed in [19]. This concept uses the continuous super-twisting controller (see [20]). A modified version of the super-twisting controller is also used in the step-by-step observer by [18]. Unfortunately, also those observers are not free of drawbacks:

1. The super-twisting algorithm provides the best-possible asymptotic accuracy of the derivative estimation at each single realization step (see [21]). In particular, with discrete measurements the accuracy is proportional to the sampling step τ in the absence of noises, and to the square root of the input noise magnitude, if the above discretization error is negligible. The step-by-step and hierarchical observers use the output of the super-twisting algorithm as noisy input at the next step. As a result, the overall observation accuracy is of the order $\tau^{\frac{1}{2^{r-1}}}$, where r is the observability index of the system. This means, for example, that in order to implement the fourth-order derivative observer with the 0.1 precision, and the unknown fifth derivative being less than 1 in its absolute value, the practically-impossible discretization step $\tau = 10^{-8}$ is needed.
2. Similarly, in the presence of the measurement noise with magnitude ε the estimation accuracy is proportional to $\varepsilon^{\frac{1}{2^r}}$, which requires measurement noises not-exceeding 10^{-16} for the fourth-order observer implementation under the above conditions.
3. The step-by-step observers [18] provide for semiglobal finite-time stability only, restricting the application of these observers to the class of the systems for which the upper bound of the initial conditions might be estimated in advance. Moreover, it works only under conditions of full relative degree, i.e. that the sum of the relative degrees of the outputs with respect to the unknown inputs equals to the dimension of the system.

At the same time the rth-order robust exact sliding-mode-based differentiator [22] removes the first issue providing for the rth derivative accuracy proportional to the discretization step τ, and resolves the second one providing for the accuracy $\varepsilon^{\frac{1}{r+1}}$. Unfortunately, its straight-forward application requires the boundedness of the unknown $(r+1)$th derivative. In practice it means that still only semiglobal observation of stable linear systems is allowed.

The High-Order Sliding-Mode observers recently developed in [23], [24], [25] provides for the **global** finite-time convergence to zero of the estimation error in strongly observable case and for the best possible accuracy. However, the application of that observer is confined to the class of the systems having a **well defined vector relative degree** with respect to the unknown inputs, i.e. a special matrix of high-order partial derivatives should be nonsingular. It turns that this is just the restriction of transformation method suggested in the above cited papers.

To avoid that restriction the technique of weakly observable subspaces and corresponding Molinari transformations [26] is proposed in [27], [25].

In section 2 we discuss the problem statement and the main notions. The algorithms for observation of strongly observable systems, unknown input identification and fault detection are presented in section 3. Section 4 contains an example illustrating proposed algorithms. Possible generalization of the obtained results and bibliographical review are considered in section 5.

2 Problem Statement and Main Notions

2.1 System Description

Consider a Linear Time-Invariant System with Unknown Inputs (LTISUI)

$$\begin{aligned} \dot{x} &= Ax + Bu(t) + E\zeta(t), \\ y &= Cx + Du(t) + F\zeta(t), \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ are the system states, $y \in \mathcal{Y} \subseteq \mathbb{R}^p$ is the vector of the system outputs, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{q_0}$ is a vector control input, $\zeta(t) \in \mathcal{W} \subseteq \mathbb{R}^m$, $m \leq p$, are the unknown inputs (disturbances or system nonlinearities), and the known matrixes A, B, C, D, E, F have suitable dimensions. The equations are understood in the Filippov sense [28] in order to provide for possibility to use discontinuous signals in controls and observers. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous. It is assumed also that all considered inputs allow the existence of solutions and their extension to the whole semi-axis $t \geq 0$.

Without loss of generality it is assumed that

$$rank \begin{bmatrix} E \\ F \end{bmatrix} = m.$$

The task is to build an observer providing the exact (preferably finite-time convergent) estimation of the states and the unknown input. Obviously, it can be assumed without loss of generality that the known input $u(t)$ is equal to zero (i.e., $u(t) = 0$).

The following notation is used in the paper. Let $G \in \mathbb{R}^{n \times m}$ be a matrix. If $\operatorname{rank} G = n$, then define the right-side pseudoinverse of G as the matrix $G^+ = G^T(GG^T)^{-1}$. If $\operatorname{rank} G = m$, then define the left-side pseudoinverse of G as the matrix $G^+ = (G^T G)^{-1} G^T$. For a matrix $J \in \mathbb{R}^{n \times m}$, $n \geq m$, with $\operatorname{rank} J = r$, we define one of the matrixes $J^\perp \in \mathbb{R}^{n-r \times n}$, such that $\operatorname{rank} J^\perp = n - r$ and $J^\perp J = 0$. The notation $J^{\perp\perp} \in \mathbb{R}^{r \times n}$ corresponds to one of the matrixes such that $\operatorname{rank} J^{\perp\perp} = r$ and $J^\perp (J^{\perp\perp})^T = 0$. It is obvious that

$$rank \begin{bmatrix} J^\perp \\ J^{\perp\perp} \end{bmatrix} = n.$$

2.2 Strong Observability, Strong Detectability and Some Their Properties

Conditions for observability and detectability of LTISUI are studied, for example, in [29], [26], [30], [31]. Recall some necessary and sufficient conditions for strong observability and strong detectability. It is assumed in the following definitions that $u(t) = 0$.

Definition 1. *([31]). The Rosenbrock matrix $R(s)$ of the quadruple $\{A, E, C, F\}$ is given by*

$$R(s) = \begin{bmatrix} sI - A & -E \\ C & F \end{bmatrix}. \tag{2}$$

The values of $s_0 \in \mathbb{C}$ such that $\operatorname{rank} R(s_0) < n + \operatorname{rank} \begin{bmatrix} -E \\ F \end{bmatrix}$ *are called invariant zeros of the quadruple $\{A, E, C, F\}$.*

Lemma 1. *([31]). Let $s_0 \in \mathbb{C}$ be an invariant zero of the quadruple $\{A, E, C, F\}$. Suppose that the initial values $x_0 \in \mathcal{X}$ and $\zeta_0 \in \mathcal{W}$ are such that*

$$R(s_0) \begin{bmatrix} x_0 \\ \zeta_0 \end{bmatrix} = 0,$$

and let the "unknown" input satisfy $\zeta(t) = e^{s_0 t}\zeta_0$. Then the corresponding output $y(t)$ is identically zero for all $t \geq 0$.

Definition 2. *([30]). System (1) is called strongly observable, if for any initial state $x(0)$ and any unknown input $\zeta(t)$, $y(t) \equiv 0$ for all $t \geq 0$ implies that also $x(t) = 0$ for all $t \geq 0$.*

2.3 The Weakly Unobservable Subspace and Its Properties

The concepts introduced in this section are further used for the development of observers.

Definition 3. *([29].) A set $\mathcal{V}$ is called A-invariant if*

$$A\mathcal{V} \subset \mathcal{V}.$$

Definition 4. *([29]). A set $\mathcal{V}$ is called (A, E)-invariant if*

$$A\mathcal{V} \subset \mathcal{V} \oplus \mathcal{E},$$

where $\mathcal{E}$ is the range space (image) of E.

Let now define three important subsets.

Definition 5. *([29]). The unobservable subspace of the pair $\{A, C\}$ is the set*

$$\mathcal{N} = \bigcap_{i=0}^{n-1} \ker\left(CA^i\right).$$

Definition 6. *([26]). A subspace $\mathcal{V}$ is called a null-output (A, E)-invariant subspace if for every $x \in \mathcal{V}$ there exists some input ζ such that $(Ax + E\zeta) \in \mathcal{V}$ and $(Cx + F\zeta) = 0$. The maximal null-output (A, E)-invariant subspace, is denoted by $\mathcal{V}^*$.*

Definition 7. *([31]). System (1) is called weakly unobservable at $x_0 \in \mathcal{X}$ if there exists an input function $\zeta(t)$, such that the corresponding output $y(t)$ equals zero for all $t \geq 0$. The set of all weakly unobservable points of (1) is denoted by $\mathcal{V}^*$ and is called the weakly unobservable subspace of (1).*

Definitions 6 and 7 actually define the same subspace. Thus, the maximal null-output (A, E)-invariant subspace and the weakly unobservable subspace coincide.

Obviously $A\mathcal{N} \subset \mathcal{N}$ and $\mathcal{N} \subset \ker(C)$. It follows from definition 6 that the weakly unobservable subspace satisfies the inclusions

$$A\mathcal{V}^* \subset \mathcal{V}^* \oplus \mathcal{E}, \quad C\mathcal{V}^* \subset \mathcal{F}. \tag{3}$$

Due to (3) the unobservable subspace of the pair (A, C) is a subset contained in the weakly unobservable subspace of (A, E, C, F), and $\mathcal{N} \subseteq \mathcal{V}^*$.

Theorem 1. *([31]). The following statements are equivalent:*

(i) The quadruple $\{A, E, C, F\}$ is strongly observable.
(ii) The quadruple $\{A, E, C, F\}$ has no invariant zeros.
(iii) The weakly unobservable subspace contains only the origin, $\mathcal{V}^* = \{0\}$.

The goal is now to design a sliding mode observer ensuring finite time observation of the states in the strongly observable case.

3 Observation of Strongly Observable LTISUI

Assumption 1 *The quadruple $\{A, C, E, F\}$ is strongly observable.*

3.1 Output Transformation

Suppose a matrix $F^{\perp}$ is selected in the form

$$F^{\perp} = \begin{bmatrix} F_{\perp}^{1} \\ F_{\perp}^{2} \end{bmatrix},$$

such that $F_{\perp}^{1} \in \mathbb{R}^{p_1 \times p}$, $F_{\perp}^{2} \in \mathbb{R}^{p_2 \times p}$, and

$$\begin{aligned}
&F_{\perp}^{1}F = 0, \text{ and } F_{\perp}^{1}CA^{i}E = 0, \, \forall\, i = 0, ..., n-1, \\
&F_{\perp}^{2}F = 0, \text{ and } F_{\perp}^{2}CA^{j-1}E \neq 0, \text{ for some } 0 \leq j < n, \\
&rank \begin{bmatrix} F_{\perp}^{1} \\ F_{\perp}^{2} \end{bmatrix} = p - p_3, \text{ and } rank F = p_3.
\end{aligned}$$

Notice that $F^{\perp}$ always can be decomposed in this form. Choose a matrix $F^{\perp\perp}$, and apply the output transformation

$$\begin{bmatrix} F^{\perp} \\ F^{\perp\perp} \end{bmatrix} y(t),$$

where $F^{\perp\perp} \in \mathbb{R}^{p_3 \times p}$. The transformed output takes the form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} x + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ F_3 \end{bmatrix} \zeta, \tag{4}$$

Note that $rank\, F_3 = rank\, F = p_3$.

Definition 8. *Consider the system (1). Define the vector of* ***partial relative degrees*** *of the output $y(t)$ with respect to the unknown vector input $\zeta(t)$ as the vector $(r_1, ..., r_p)$ composed of the integers r_i, $i = 1, ..., p$. Each partial relative degree r_i satisfies the following requirements:*

- *$r_i = 0$, if $f_{3_i} \neq 0$, where f_{3_i} is the ith row of the matrix F_3;*
- *If $f_{3_i} = 0$, then r_i is the integer such that*

$$\begin{aligned} &c_i A^j E = 0,\ j = 0, ..., r_i - 2, \quad c_i A^{r_i - 1} E \neq 0, \\ &r_i \leq n - 1 \end{aligned} \tag{5}$$

  where c_i is the ith row of the matrix C;
- *and $r_i = \infty$, if $f_{3_i} = 0$ and $c_i A^j E = 0$ for all $j = 0, ..., n - 1$.*

In other words $F_{\perp}^{1} y$ corresponds to the outputs with partial relative degree equal to infinity, and $F_{\perp}^{2} y$ corresponds to the outputs with finite-positive partial relative degree. The vector $y_1(t) \in \mathbb{R}^{p_1}$ is composed of all the outputs with partial relative degree equal to ∞, the components of $y_2(t) \in \mathbb{R}^{p_2}$ correspond to the outputs with finite partial relative degree such that $0 < r_i < n - 1$ for $i = 1, ..., p_2$, and the output $y_3(t) \in \mathbb{R}^{p_3}$ is composed by all the outputs with partial relative degree equal to 0 with respect to the unknown inputs.

Remark 1. The standard definition of the vector relative degree [32] requires the non-singularity of a specific matrix. The introduced notion removes this restriction.

3.2 State Transformation

Consider the system output (4) and the first equation of (1). Now we will separate the state dynamics contaminated by the unknown inputs and the "clean" state dynamics.

Define n_{y_1} as the rank of the observability matrix of the pair (C_1, A) (see [33]). Let the matrix $U_{y_1} \in \mathbb{R}^{n_{y_1} \times n}$ be composed by the first n_{y_1} linearly independent rows of the observability matrix. The matrix U_{y_1} is further called the reduced order observability matrix of the pair (C_1, A).

The observable subspace of the pair (C_1, A) is free from the unknown input. Choose one of the matrixes $\bar{U}_{y_1} \in \mathbb{R}^{(n-n_{y_1})\times n}$ so that

$$U_{y_1}\bar{U}_{y_1}^T = 0, \qquad \operatorname{rank}\begin{bmatrix} U_{y_1} \\ \bar{U}_{y_1} \end{bmatrix} = n.$$

and define the transformation matrix

$$T_y = \begin{bmatrix} U_{y_1} \\ \bar{U}_{y_1} \end{bmatrix}. \tag{6}$$

The system (1) with the transformed outputs (4) could be written in the equivalent form

$$\begin{aligned} \dot{x} &= Ax + Bu + E\zeta \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} x + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ F_3 \end{bmatrix} \zeta \end{aligned} \tag{7}$$

Theorem 2. *Consider the state transformation $\xi = T_y x$ with T_y defined by (6). The system (7) is transformed into the form*

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} \zeta, \tag{8}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ F_3 \end{bmatrix} \zeta, \tag{9}$$

where $\xi_1 \in \mathbb{R}^{n_{y_1}}$, $\xi_2 \in \mathbb{R}^{(n-n_{y_1})}$.

See the corresponding proof in the appendix.

Corollary 1. *The subsystem of (8), (9), describing the dynamics of $\xi_1 \in \mathbb{R}^{n_{y_1}}$*

$$\begin{aligned} \dot{\xi}_1 &= A_{11}\xi_1 + B_1 u, \\ y_1 &= C_{11}\xi_1 + D_1 u, \end{aligned}, \tag{10}$$

is observable.

The proof of this corollary is given in the appendix.

3.3 Observer Design

Assumption 2. *The unknown input $\zeta(t)$ is a Lebesgue-measurable function and is bounded, i. e. $||\zeta(t)|| \leq \zeta^+$.*

The observer is designed in two steps. First, the convergence of the estimation error to a bounded vicinity of the origin is ensured. Second, the bounded estimation error is forced to vanish using a differentiator based on high-order sliding modes.

3.4 Bounding the Estimation Error

Note that the eigenvalues of the matrix A from (1) are the union of the set of eigenvalues of the matrixes A_{11}, A_{22} from (8).

Consider the system (8), (9). Select a gain matrix

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & L_{23} \end{bmatrix} \in \mathbb{R}^{n \times p},$$

where $L_{11} \in \mathbb{R}^{n_{y_1} \times p_1}$, $L_{22} \in \mathbb{R}^{(n-n_{y_1}) \times p_2}$, $L_{23} \in \mathbb{R}^{(n-n_{y_1}) \times p_3}$, so that $A_{11} - L_{11}C_{11}$, $A_{22} - L_{22}C_{22} - L_{23}C_{32}$ be Hurwitz. The gain matrix L exists due the assumption 3.

The Luenberger part of the observer takes on the form

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ &+ \begin{bmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & L_{23} \end{bmatrix} (y - \hat{y}), \end{aligned} \tag{11}$$

where $\hat{y}(t)$ is the output estimation

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} u. \tag{12}$$

The corresponding error system is

$$\begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{E}_2 \end{bmatrix} \zeta(t), \tag{13}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_3 \end{bmatrix} \zeta(t), \tag{14}$$

where $\tilde{e} = \xi - z$, $\tilde{y} = y - \hat{y}$, and the matrixes $\tilde{A}_{11}$, $\tilde{A}_{21}$, $\tilde{A}_{22}$, $\tilde{E}_2$ are defined as

$$\begin{aligned} \tilde{A}_{11} &= A_{11} - L_{11}C_{11}, \quad \tilde{A}_{21} = A_{21} - L_{23}C_{31}, \\ \tilde{A}_{22} &= A_{22} - L_{22}C_{22} - L_{23}C_{32}, \quad \tilde{E}_2 = L_{23}F_3 + E_2. \end{aligned}$$

The equations (13) and (14) can be rewritten in a compact form as

$$\dot{\tilde{e}} = \tilde{A}\tilde{e} + \tilde{E}\zeta(t), \tag{15}$$

$$\tilde{y} = \tilde{C}\tilde{e} + \tilde{F}\zeta(t). \tag{16}$$

3.5 Finite Time Convergence Enforcement

In this subsection concerns the estimation of certain number of derivatives of the outputs y_1 and y_2 by the robust-exact differentiator [22] and the linear combination of this derivatives with the output y_3 to reconstruct the state coordinates.

Denote

$$\tilde{C} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \\ C_{31} & C_{32} \end{bmatrix} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{bmatrix}.$$

Consider the error estimation system (15), (16). Obtain the matrixes

$$U_{1i}^T = \left[\, \tilde{c}_{1i}^T \ (\tilde{c}_{1i}\tilde{A})^T \ \cdots \ (\tilde{c}_{1i}\tilde{A}^{n_{1i}-1})^T \,\right]$$

where $\tilde{c}_{1i}$, $i = 1, ..., p_1$ is the ith row of the matrix $\tilde{C}_1$, and n_{1i} is the integer defined as

$$n_{1i} = rank\ \left[\, \tilde{c}_{1i}^T \ (\tilde{c}_{1i}\tilde{A})^T \ \cdots \ (\tilde{c}_{1i}\tilde{A}^{n-1})^T \,\right]^T.$$

It is easy to see that the matrix U_{1i} is the observability matrix for the pair $(\tilde{c}_{1i}, \tilde{A})$.

Surely, the output-estimation error $\tilde{y}_1$ is measurable. Apply the differentiator by [22] to each component of $\tilde{y}_1$:

$$\begin{aligned}
\dot{v}_{i1} &= w_{i1} = -\alpha_{n_{1i}} N_i^{1/n_{1i}} |v_{i1} - \tilde{y}_{1i}|^{(n_{1i}-1)/n_{1i}} \operatorname{sign}(v_{i1} - \tilde{y}_{1i}) + v_{i2}, \\
\dot{v}_{i2} &= w_{i2} = -\alpha_{(n_{1i}-1)} N_i^{1/(n_{1i}-1)} \times \\
&\qquad |v_{i2} - w_{i1}|^{(n_{1i}-2)/(n_{1i}-1)} \operatorname{sign}(v_{i2} - w_{i1}) + v_{i3}, \\
&\ \ \vdots \\
\dot{v}_{i,n_{1i}-1} &= w_{i,n_{1i}-1} = -\alpha_2 N_i^{1/2} |v_{i,n_{1i}-1} - w_{i,n_{1i}-2}|^{1/2} \times \\
&\qquad \operatorname{sign}(v_{i,n_{1i}-1} - w_{i,n_{1i}-2}) + v_{in_{1i}}, \\
\dot{v}_{in_{1i}} &= -\alpha_1 N_i \operatorname{sign}(v_{in_{1i}} - w_{i,n_{1i}-1}),
\end{aligned} \tag{17}$$

where $N_i > 0$ and the constants α_i are recursively chosen sufficiently large for all the components as in ([22]). In particular, one of the possible choices is $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, which is sufficient for $n_{1i} \leq 6$. The obtained components v_{i_j} can be arranged in the vector

$$\tilde{v}_i^T = \left[\, v_{i1}^T \ v_{i2}^T \ \cdots \ v_{in_{1i}}^T \,\right].$$

For all $\tilde{v}_i$ and U_{1i}, $i = 1, ..., p_1$, the equality $\tilde{v}_i = U_{1i}\tilde{e}$ holds after finite time.

It is possible to find the matrixes $U_{1_{\text{extended}}}$ and v_{extended} as:

$$U_{1\text{extended}} = \begin{bmatrix} U_{11} \\ U_{12} \\ \vdots \\ U_{1p_1} \end{bmatrix}, \quad v_{\text{extended}} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_{p_1} \end{bmatrix}.$$

it is clear that $rank\, U_{1\text{extended}} = n_{y_1}$. Construct the matrix $U_1 \in \mathbb{R}^{n_{y_1} \times n}$ selecting the first n_{y_1} linearly independent rows of $U_{1\text{extended}}$ and the vector v composed of the corresponding rows of the matrix v_{extended}, so that the equality $v = U_1\tilde{e}$ holds after finite time.

Compute the vector of partial relative degrees. Let $\tilde{r}_i$ be the vector of partial relative degrees of the output $\tilde{y}_{2i}$ with respect to the unknown inputs, where $\tilde{y}_{2i}$ is the ith component of the output $\tilde{y}_2$.

For every row of $\tilde{C}_2$ obtain

$$\tilde{U}_{2i}^T = \left[\tilde{c}_{2i}^T \; (\tilde{c}_{2i}\tilde{A})^T \; \cdots \; (\tilde{c}_{2i}\tilde{A}^{\tilde{r}_i - 1})^T \right],$$

where $\tilde{c}_{2i}$, $i = 1, \dots p_2$ is the ith row of the matrix $\tilde{C}_2$, and $\tilde{r}_i$ is the corresponding vector of partial relative degrees of the ith component of the output-estimation error $\tilde{y}_2$ with respect to the unknown inputs.

Note that

$$\begin{bmatrix} \tilde{y}_{2i} \\ \dot{\tilde{y}}_{2i} \\ \vdots \\ \tilde{y}_{2i}^{(\tilde{r}_i - 1)} \end{bmatrix} = \begin{bmatrix} \tilde{c}_{2i} \\ \tilde{c}_{2i}\tilde{A} \\ \vdots \\ \tilde{c}_{2i}\tilde{A}^{\tilde{r}_i - 1} \end{bmatrix} \tilde{e} \tag{18}$$

where $\tilde{y}_{2i}$, $i = 1 \dots, p_2$ is the ith row of $\tilde{y}_2$, and $\tilde{y}_{2i}^{(k)}$ denotes the kth derivative of $\tilde{y}_{2i}$.

Apply the high order sliding mode differentiator by [22] to each component of $\tilde{y}_2$ as

$$\begin{aligned}
\dot{\bar{v}}_{i1} &= \bar{w}_{i1} = -\alpha_{\tilde{r}_i}\bar{N}_i^{1/\tilde{r}_i}|\bar{v}_{i1} - \tilde{y}_{2i}|^{(\tilde{r}_i - 1)/\tilde{r}_i}\operatorname{sign}(\bar{v}_{i1} - \tilde{y}_{2i}) + \bar{v}_{i2}, \\
\dot{\bar{v}}_{i2} &= \bar{w}_{i2} = -\alpha_{\tilde{r}_i - 1}\bar{N}_i^{1/(\tilde{r}_i - 1)}|\bar{v}_{i2} - \bar{w}_{i1}|^{(\tilde{r}_i - 2)/(\tilde{r}_i - 1)}\operatorname{sign}(\bar{v}_{i2} - \bar{w}_{i1}) + \bar{v}_{i3}, \\
&\;\vdots \\
\dot{\bar{v}}_{i,\tilde{r}_i - 1} &= \bar{w}_{i,\tilde{r}_i - 1} = -\alpha_2\bar{N}_i^{1/2}|\bar{v}_{i,\tilde{r}_i - 1} - \bar{w}_{i,\tilde{r}_i - 2}|^{1/2}\operatorname{sign}(\bar{v}_{i,\tilde{r}_i - 1} - \bar{w}_{i,\tilde{r}_i - 2}) + \bar{v}_{i\tilde{r}_i}, \\
\dot{\bar{v}}_{i\tilde{r}_i} &= -\alpha_1\bar{N}_i\operatorname{sign}(\bar{v}_{i\tilde{r}_i} - \bar{w}_{i,\tilde{r}_i - 1}),
\end{aligned} \tag{19}$$

where $\bar{v}_{i_j}$ and $\bar{w}_{i_j}$ are the components of the vectors $\bar{v}_i \in \mathbb{R}^{\tilde{r}_i}$ and $\bar{w}_i \in \mathbb{R}^{\tilde{r}_i - 1}$ respectively. The parameter $\bar{N}_i$ is chosen sufficiently large for each output estimation error, in particular, $\bar{N}_i > |d_i|\zeta^+$ is required, where $d_i = \tilde{c}_{2i}\tilde{A}^{\tilde{r}_i - 1}\tilde{E}$. The constants α_i are chosen recursively sufficiently large for all the components as in [22]. In particular, one of the possible choices is $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, which is sufficient for $\tilde{r}_i \leq 6$. Note that (19) has a recursive form, useful for the parameter tuning as was given in [22].

For each component of $\tilde{y}_2$ form the vector

$$\bar{v}_i^T = \left[\bar{v}_{i1}^T \; \bar{v}_{i2}^T \; \cdots \; \bar{v}_{1,\tilde{r}_i}^T \right].$$

Note that the vector $\bar{v}_i$ in finite time satisfies the relation

$$\begin{bmatrix} \tilde{y}_{2i} \\ \dot{\tilde{y}}_{2i} \\ \vdots \\ \tilde{y}_{2i}^{(\tilde{r}_i - 1)} \end{bmatrix} = \begin{bmatrix} \bar{v}_{i1} \\ \bar{v}_{i2} \\ \vdots \\ \bar{v}_{i\tilde{r}_i} \end{bmatrix}.$$

Define the extended matrix $U_{2\text{extended}}$, the extended vector $\bar{v}_{\text{extended}}$, and compute the integer n_{o2} as

$$\begin{aligned}\bar{v}_{\text{extended}} &= \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{p_2} \end{bmatrix}, \quad U_{2\text{extended}} = \begin{bmatrix} \tilde{U}_{21} \\ \tilde{U}_{22} \\ \vdots \\ \tilde{U}_{2p_2} \end{bmatrix}, \\ n_{o2} &= rank(U_{2\text{extended}}).\end{aligned} \tag{20}$$

Take the full row rank matrix $U_2 \in \mathbb{R}^{n_{o2}\times n}$ composed by the first n_{o2} linearly independent rows of the matrix $U_{2_{\text{extended}}}$, and select the corresponding rows of $\bar{v}_{\text{extended}}$ so that the equality $\bar{v} = U_2\tilde{e}$ holds after finite time.

Consider the derivatives of order $\tilde{r}_i - 1$ of each component of $\tilde{y}_2$. The following equality holds:

$$\begin{bmatrix} \tilde{y}_{2_1}^{\tilde{r}_1-1} \\ \tilde{y}_{2_2}^{\tilde{r}_2-1} \\ \vdots \\ \tilde{y}_{2_{p_2}}^{\tilde{r}_{p_2}-1} \end{bmatrix} = \begin{bmatrix} \tilde{c}_{2_1}\tilde{A}^{\tilde{r}_1-1} \\ \tilde{c}_{2_2}\tilde{A}^{\tilde{r}_2-1} \\ \vdots \\ \tilde{c}_{2_{p_2}}\tilde{A}^{\tilde{r}_{p_2}-1} \end{bmatrix}\tilde{e}.$$

Define the matrixes

$$\bar{U}_r = \begin{bmatrix} \tilde{c}_{2_1}\tilde{A}^{\tilde{r}_1-1} \\ \tilde{c}_{2_2}\tilde{A}^{\tilde{r}_2-1} \\ \vdots \\ \tilde{c}_{2_{p_2}}\tilde{A}^{\tilde{r}_{p_2}-1} \end{bmatrix}, \quad \bar{v}_r = \begin{bmatrix} \bar{v}_{1_{\tilde{r}_1}} \\ \bar{v}_{2_{\tilde{r}_2}} \\ \vdots \\ \bar{v}_{p_{2_{\tilde{r}_{p_2}}}} \end{bmatrix}.$$

Make $\tilde{M}_0 = 0_{n\times n}$ and $\rho_0 = 0_{n\times 1}$. Define the matrix $\tilde{\rho}_i$ as

$$\tilde{\rho}_i = \begin{bmatrix} \tilde{M}_i\tilde{E} \\ \bar{U}_r\tilde{E} \\ F_3 \end{bmatrix}^{\perp} \begin{bmatrix} \rho_i \\ \bar{v}_r \\ \int_0^t y_3 dt \end{bmatrix}$$

For each component of the vector $\tilde{\rho}_i$ compute

$$\begin{aligned}\dot{\beta}_{ij} &= \alpha_2\bar{N}_i^{1/2}|\beta_{ij} - \tilde{\rho}_{ij}|^{1/2} sign(\beta_{ij} - \tilde{\rho}_{ij}) + \gamma_{ij} \\ \dot{\gamma}_{ij} &= \alpha_1\bar{N}_i sign(\gamma_{ij} - \dot{\beta}_{ij})\end{aligned} \tag{21}$$

where $\tilde{\rho}_{i_j}$ is the jth component of the vector $\tilde{\rho}_i$; $\bar{N}_i > ||\bar{D}_i||\zeta^+$, where $\bar{D}_i$ is the ith row of the matrix

$$\bar{D} = \begin{bmatrix} \tilde{M}_i\tilde{E} \\ \bar{U}_r\tilde{E} \\ F_3 \end{bmatrix}^{\perp} \tilde{A}\tilde{E}.$$

Notice that the matrix $\bar{D}$ is computed for each matrix $\tilde{M}_i$ to appear below.

Compute the matrix $\tilde{M}_{i+1}$ and the vector ρ_{i+1}:

$$\tilde{M}_{i+1} = \begin{bmatrix} \tilde{M}_i \tilde{E} \\ \bar{U}_r \tilde{E} \\ F_3 \end{bmatrix}^{\perp} \begin{bmatrix} \tilde{M}_i \tilde{A} \\ \bar{U}_r \tilde{A} \\ C_3 \end{bmatrix}, \quad \rho_{i+1} = \begin{bmatrix} \gamma_{i1} \\ \gamma_{i2} \\ \vdots \\ \gamma_{i\kappa} \end{bmatrix} \tag{22}$$

Here

$$\kappa = \text{rank} \begin{bmatrix} \tilde{M}_i \tilde{E} \\ \bar{U}_r \tilde{E} \\ F_3 \end{bmatrix}^{\perp}.$$

This computation is performed until

$$rank \begin{bmatrix} U_1 \\ U_2 \\ \tilde{M}_{i+1} \end{bmatrix} = n.$$

is satisfied. Let l be the number of computed matrices M_i. Select the first $n - n_{y_1} - n_{o2}$ linearly independent rows of $\tilde{M}_{i+1}$ to form the matrix M_d such that

$$rank \begin{bmatrix} U_1 \\ U_2 \\ \tilde{M}_d \end{bmatrix} = n,$$

and select the corresponding components of ρ_{i+1} to form the vector ρ_d. Compute the matrix M_n and the vector ρ_n as

$$M_n = \begin{bmatrix} U_2 \\ \tilde{M}_d \end{bmatrix}, \quad \rho_n = \begin{bmatrix} \bar{v} \\ \rho_d \end{bmatrix}.$$

It is clear that the equality

$$\rho_n = M_n \tilde{e}.$$

holds after finite time. The algorithm with finite time convergence of the estimation error is given by

$$\hat{\xi} = z + \begin{bmatrix} U_1 \\ M_n \end{bmatrix}^{-1} \begin{bmatrix} v \\ \rho_n \end{bmatrix}. \tag{23}$$

Theorem 3. *Let assumptions 3 and 3.3 be satisfied. The state of the system (8) is estimated* ***exactly*** *and in* ***finite time*** *by the observer (11), (12), (19), (21), (22), (23).*

Proof. Prove that the application of (11), (12) ensures the convergence of the estimation error (15), (16) to a bounded vicinity of the origin.

If the matrix $\tilde{A}$ is Hurwitz, then also the matrices $\tilde{A}_{11}$, $\tilde{A}_{22}$ are Hurwitz.

Choose the Lyapunov function of the system

$$V = \tilde{e}^T H \tilde{e},$$

where H is a symmetric positive-definite matrix. The matrix H is chosen as the solution of the Lyapunov equation

$$H\tilde{A} + \tilde{A}^T H = -I.$$

It is used here that $\tilde{A}$ is a Hurwitz matrix. Calculate the derivative

$$\begin{array}{l}\dot{V} = \tilde{e}^T(H\tilde{A} + \tilde{A}^T H)\tilde{e} + (\tilde{e}^T H\tilde{E}\zeta(t) + \zeta^T(t)\tilde{E}^T H\tilde{e}) \\ \dot{V} \leq -I||\tilde{e}||^2 + 2||\tilde{e}||\,||H||\,||\tilde{E}||\zeta^+.\end{array}$$

$\dot{V}$ is negative definite with $\zeta(t) = 0$. Thus, if ζ satisfies the assumption 3.3, obtain that the estimation error $\tilde{e}$ converges to a bounded vicinity of the origin $\tilde{e} = 0$. Since that moment also $\dot{\tilde{e}}$ remains uniformly bounded.

Now consider subsystem (10). The application of the observer (11), (12) produces the estimation error

$$\begin{array}{l}\dot{\tilde{e}}_1 = \tilde{A}_{11}\tilde{e}_1 \\ \tilde{y}_1 = C_{11}\tilde{e}_1\end{array}. \tag{24}$$

Since $\tilde{A}_{11}$ is Hurwitz, the estimation error $\tilde{e}_1 = \xi_1 - z_1$ asymptotically converges to zero.

Define the estimation error as $e = \xi - \hat{\xi}$, then from equation (23) obtain

$$e = \xi - z - \begin{bmatrix} U_1 \\ M_n \end{bmatrix}^{-1} \begin{bmatrix} v \\ \rho_n \end{bmatrix} = \tilde{e} - \begin{bmatrix} U_1 \\ M_n \end{bmatrix}^{-1} \begin{bmatrix} v \\ \rho_n \end{bmatrix}$$

Now multiply the last equation by the matrix $\begin{bmatrix} U_1 \\ M_n \end{bmatrix}$:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ M_n \end{bmatrix} \tilde{e} - \begin{bmatrix} v \\ \rho_n \end{bmatrix} = \begin{bmatrix} U_1 \\ M_n \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} - \begin{bmatrix} v \\ \rho_n \end{bmatrix}. \tag{25}$$

Prove the convergence of e_1 to zero. Note that by definition $U_1 \begin{bmatrix} 0 \\ \tilde{e}_2 \end{bmatrix} = 0$, and consequently e_1 only depends on $\tilde{e}_1$.

Consider the HOSM-differentiator (17). Prove that the equality $v_{ij} = \tilde{y}_{1i}^{(j-1)}$ holds for each $j = 1, ..., n_{1i}$, $i = 1, ..., p_1$.

Denote the sliding variables $\sigma_{ij} = v_{ij} - (\tilde{y}_{1i})^{(j-1)}$ and obtain

$$\begin{array}{l}\dot{\sigma}_{i1} = -\alpha_{n_{1i}} N_i^{1/n_{1i}} |\sigma_{i1}|^{(n_{1i}-1)/n_{1i}} \operatorname{sign}(\sigma_{i1}) + \sigma_{i2}, \\ \dot{\sigma}_{i2} = -\alpha_{n_{1i}-1} N_i^{1/(n_{1i}-1)} |\sigma_{i2} - \dot{\sigma}_{i1}|^{(n_{1i}-2)/(n_{1i}-1)} \operatorname{sign}(\sigma_{i2} - \dot{\sigma}_{i1}) + \sigma_{i3}, \\ \quad \vdots \\ \dot{\sigma}_{i,n_{1i}-1} = -\alpha_2 N_i^{1/2} |\sigma_{i,n_{1i}-1} - \dot{\sigma}_{i,n_{1i}-2}|^{1/2} \operatorname{sign}(\sigma_{i,n_{1i}-1} - \dot{\sigma}_{i,n_{1i}-2}) + \sigma_{i,n_{1i}}, \\ \dot{\sigma}_{i,n_{1i}} = -\alpha_1 N_i \operatorname{sign}(\sigma_{i,n_{1i}} - \dot{\sigma}_{i,n_{1i}-1}) - \tilde{y}_{1i}^{(n_{1i})}.\end{array} \tag{26}$$

Show now that the dynamics of σ_{ij} is finite-time stable. Since (24) is stable, starting from some moment, $\tilde{e}_i$ and $\dot{\tilde{e}}_i$ remain inside a bounded zone with the maximal amplitude N_i. The dynamics (26) satisfies the differential inclusion

$$\begin{aligned}
&\dot{\sigma}_{i1} = -\alpha_{n_{1i}} N_i^{1/n_{1i}} |\sigma_{i1}|^{(n_{1i}-1)/n_{1i}} \operatorname{sign}(\sigma_{i1}) + \sigma_{i2},\\
&\dot{\sigma}_{i2} = -\alpha_{n_{1i}-1} N_i^{1/(n_{1i}-1)} |\sigma_{i2} - \dot{\sigma}_{i1}|^{(n_{1i}-2)/(n_{1i}-1)} \operatorname{sign}(\sigma_{i2} - \dot{\sigma}_{i1}) + \sigma_{i3},\\
&\quad\vdots\\
&\dot{\sigma}_{i_{(n_{1i}-1)}} = -\alpha_2 N_i^{1/2} |\sigma_{i,n_{1i}-1} - \dot{\sigma}_{i,n_{1_i}-2}|^{1/2} \operatorname{sign}(\sigma_{i,n_{1i}-1} - \dot{\sigma}_{i,n_{1i}-2}) + \sigma_{i,n_{1i}},\\
&\dot{\sigma}_{i,n_{1_i}} \in -\alpha_1 N_i \operatorname{sign}(\sigma_{i,n_{1i}} - \dot{\sigma}_{i,n_{1i}-1}) + [-N_i\ N_i].
\end{aligned} \tag{27}$$

The rest of the proof is based on the following Lemma.

Lemma 2. *Suppose that $\alpha_1 > 1$ and α_2, ..., $\alpha_{n_{1i}}$ are chosen sufficiently large in the list order. Then after finite time of the transient process any solution of (27) satisfies the equalities $|\sigma_{ij}| = 0$, $j = 1, 2, ..., n_{1i}$.*

See the proof in the appendix.

Thus, there exists a high order sliding mode $\sigma_{i1} = ... = \sigma_{i,n_{1i}} = 0$ and after finite time the equality

$$\begin{bmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_{n_{1_i}}} \end{bmatrix} = \begin{bmatrix} \tilde{y}_{1_i} \\ \dot{\tilde{y}}_{1_i} \\ \vdots \\ \tilde{y}_{1_i}^{(n_{1_i}-1)} \end{bmatrix} \tag{28}$$

is kept. The matrixes U_{1i} can be written as

$$U_{1i} = \begin{bmatrix} c_{11i} & 0 \\ c_{11i}\tilde{A}_{11} & 0 \\ \vdots & \vdots \\ c_{11i}\tilde{A}_{11}^{n_{1i}-1} & 0 \end{bmatrix},$$

and the product $U_{1i}\tilde{e}$ can be expressed as

$$U_{1i}\tilde{e} = \begin{bmatrix} c_{11i} \\ c_{11i}\tilde{A}_{11} \\ \vdots \\ c_{11i}\tilde{A}_{11}^{n_{1i}-1} \end{bmatrix} \tilde{e}_1$$

Note that on the other hand the following equality holds:

$$\begin{bmatrix} \tilde{y}_{1i} \\ \dot{\tilde{y}}_{1i} \\ \vdots \\ \tilde{y}_{1i}^{(n_{1i}-1)} \end{bmatrix} = \begin{bmatrix} c_{11i} \\ c_{11i}\tilde{A}_{11} \\ \vdots \\ c_{11i}\tilde{A}_{11}^{n_{1i}-1} \end{bmatrix} \tilde{e}_1, \tag{29}$$

where $\tilde{y}_{1i}^{(n_{1i}-1)}$, $i = 1, ..., p_1$ denotes the derivative of order $n_{1i} - 1$ of the ith component of the vector $\tilde{y}_1$.

The component e_1 of the estimation error is expressed as

$$e_1 = U_1 \tilde{e} - v. \tag{30}$$

The matrix U_1 and the vector v are composed of rows of the extended matrices $\tilde{U}_{1extended}$ and $v_{extended}$. The rest of the proof is a consequence of the equalities (28) and (29).

Consider the component e_2 of the estimation error:

$$e_2 = M_n \tilde{e} - \rho_n$$

Substitute the value of M_n and ρ_n, computed according to (22), to the right hand side of the last equation and obtain

$$e_2 = \begin{bmatrix} U_2 \\ M_d \end{bmatrix} \tilde{e} - \begin{bmatrix} \bar{v} \\ \rho_d \end{bmatrix}$$

The convergence to zero of e depends on the convergence of v to $\tilde{e}$ and ρ_d to $M_d \tilde{e}$.

Convergence $\bar{v} \to U_2 \tilde{e}$. Consider the auxiliary variable $\bar{e}_{2i} = \tilde{U}_{2i} \tilde{e}$, constructed for each block of the extended matrix $\tilde{U}_2$. The equality (18) holds for each block, then the vector $\bar{e}_{2i}$ could be represented as

$$\bar{e}_{2i} = \begin{bmatrix} \tilde{y}_{2i} \\ \dot{\tilde{y}}_{2i} \\ \vdots \\ \tilde{y}_{2i}^{(\tilde{r}_i - 1)} \end{bmatrix}.$$

Now it is clear that the next step is to prove that $\tilde{v} \to \bar{e}_{2i}$.

For each $\tilde{y}_{2i}$, $i = 1, ..., p_2$ denote $\sigma_{ij} = v_{ij} - (\tilde{y}_{2i})^{(j-1)}$ and obtain similar equations to (26) and (27), so that with sufficiently large $\bar{N}_i$ the dynamics of σ_i is finite-time stable, $|d_i|\zeta^+ > |y_{2i}^{(\tilde{r}_i)}|$. Starting from some moment, $\tilde{e}$ remains uniformly bounded, and the same is true with respect to $\bar{e}$.

The convergence of the differential inclusion is a consequence of Lemma 2.

The next equality is established in finite time

$$\begin{bmatrix} \tilde{y}_{2i} \\ \dot{\tilde{y}}_{2i} \\ \vdots \\ \tilde{y}_{2i}^{(\tilde{r}_i - 1)} \end{bmatrix} = \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{i\tilde{r}_i} \end{bmatrix}.$$

The finite time convergence of $\tilde{v}$ to $\tilde{U}_2 \tilde{e}$ is ensured. The vector v and the matrix U_2 are selected as $\tilde{v}$ and $\tilde{U}_2$. Thus with the appropriate selection of v and U_2, the equality $v = U_2 \tilde{e}$ holds after finite time.

Convergence $\rho_d \to M_d \tilde{e}$. Consider the computation (22). The application of the algorithm (21) to the coordinate ρ_i could be seen as a particular case of (19) with $\tilde{r}_i = 2$. Hence the equality $\gamma_i = \dot{\rho}_i$ is established in finite time.

It was proved that $v \to U_1 \tilde{e}_1$, $\bar{v} \to U_2 \tilde{e}$, $\rho_d \to M_d \tilde{e}$. Now the finite time convergence of e_1 and e_2 is a direct consequence.

3.6 Unknown Input Identification

Let the following assumption hold.

Assumption 3. *The kth order derivative of the unknown input $\zeta_i^{(k)}(t)$ exists almost everywhere and is a bounded Lebesgue-measurable function, $|\zeta_i^{(k)}(t)| \leq \zeta_{1i}^{+}$.*

Denote

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ M_n \end{bmatrix}^{-1} \begin{bmatrix} \bar{v} \\ \rho_n \end{bmatrix} \tag{31}$$

where $\hat{e}_1 \in \mathbb{R}^{n_{y1}}$ and $\hat{e}_2 \in \mathbb{R}^{n-n_{y1}}$.

The unknown input can be identified by means of the identity

$$\hat{\zeta} = \begin{bmatrix} \tilde{E}_2 \\ F_3 \end{bmatrix}^{+} \begin{bmatrix} \dot{\hat{e}}_2 - \begin{bmatrix} \tilde{A}_{21} \ \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} \\ \tilde{y}_3 - \begin{bmatrix} C_{31} \ C_{32} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} \end{bmatrix}. \tag{32}$$

The vectors $\hat{e}_1$, $\hat{e}_2$ are known. The value of $\dot{\hat{e}}_2$ is computed in two different forms according to the properties of the unknown input, and the structure of the matrix M_n.

The value of $\dot{\hat{e}}_2$ is computed using the equality

$$\begin{bmatrix} \dot{\hat{e}}_1 \\ \dot{\hat{e}}_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ M_n \end{bmatrix}^{-1} \begin{bmatrix} \dot{\bar{v}} \\ \dot{\rho}_n \end{bmatrix}.$$

The vector $\dot{\bar{v}}$ and the vector $\dot{\rho}_n$ can be computed in two different ways. The first method is applied if the unknown input is discontinuous, and the second if the unknown input satisfies assumption 3.6.

Non smooth unknown input identification. Consider the vector

$$\dot{\bar{v}}_i = \begin{bmatrix} v_{i_2} \\ v_{i_3} \\ \vdots \\ v_{i_{n_{1_i}}} \\ \text{filtered}(\dot{v}_{i_{n_{1_i}}}) \end{bmatrix}. \tag{33}$$

The term $\dot{v}_{in_{1i}}$ is a high frequency component evaluated in (19). The high frequency component should be filtered out to obtain the component filtered$(\dot{v}_{in_{1i}})$.

Consider the last iteration applied to obtain M_n. Since the value of $\ddot{\hat{\rho}}_{ij}$ is a high frequency term, it has to be filtered to obtain an estimated value of $\ddot{\hat{\rho}}_{ij}$, to form the matrix $\dot{\rho}_n$.

Smooth unknown input identification. The second method to obtain the values of $\dot{\bar{v}}$ and $\dot{\rho}_n$ is to extend (17) from n_{1i} components to $n_{1i} + k$ components:

$$
\begin{aligned}
&\dot{v}_{i,1} = w_{i,1} = -\alpha_{r_i+k} M^{1/(r_i+k)} |v_{i,1} - \tilde{e}_y|^{(r_i+k-1)/(r_i+k)} \operatorname{sign}(v_{i,1} - \tilde{e}_y) + v_{i,2}, \\
&\dot{v}_{i,2} = w_{i,2} = -\alpha_{r_i+k-1} M^{1/(r_i+k-1)} |v_{i,2} - w_{i,1}|^{(r_i+k-2)/(r_i+k-2)} \times \\
&\quad \operatorname{sign}(v_{i,2} - w_{i,1}) + v_{i,3}, \\
&\vdots \\
&\dot{v}_{i,r_i} = w_{i,r_i} = -\alpha_{k+1} M^{1/(k+1)} |v_{i,r_i} - w_{i,r_i-1}|^{(k)/(k+1)} \times \\
&\quad \operatorname{sign}(v_{i,r_i} - w_{i,r_i-1}) + v_{r_i+1}, \\
&\vdots \\
&\dot{v}_{i,r_i+k} = -\alpha_1 M \operatorname{sign}(v_{i,r_i+k} - w_{i,r_i+k-1}),
\end{aligned}
\tag{34}
$$

Define the vector $\dot{\tilde{v}}_i$ as

$$
\dot{\tilde{v}}_i = \begin{bmatrix} v_{i_2} \\ v_{i_3} \\ \vdots \\ v_{i_{n_{1_i}+1}} \end{bmatrix}; \tag{35}
$$

and define the extended vector $\dot{\tilde{v}}_{\text{extended}} = [\dot{\tilde{v}}_1^T \quad ... \quad \dot{\tilde{v}}_{p_1}^T]^T$. Select the same rows, which have been chosen to form U_1, to form the vector $\dot{\bar{v}}$. Note that $\dot{\bar{v}} = U_1 \tilde{A} \tilde{e}_1$.

If the unknown input satisfies Hypothesis 3.6, it is possible to extend the order of (21) to the second one:

$$
\begin{aligned}
\dot{\hat{\rho}}_{ij} &= \beta_1 N_{\rho_{ij}}^{1/3} |\hat{\rho}_{ij} - \rho_{ij}|^{2/3} sign(\hat{\rho}_{ij} - \dot{\hat{\rho}}_{ij}) + \int \ddot{\hat{\rho}}_{ij} dt \\
\ddot{\hat{\rho}}_{ij} &= \beta_2 N_{\rho_{ij}}^{1/2} |\int \ddot{\hat{\rho}}_{ij} dt - \dot{\hat{\rho}}_{ij}|^{1/2} sign(\int \ddot{\hat{\rho}}_{ij} dt - \dot{\hat{\rho}}_{ij}) + \int \dddot{\hat{\rho}}_{ij} dt \\
\dddot{\hat{\rho}}_{ij} &= \beta_3 N_{\rho_{ij}} sign(\int \dddot{\hat{\rho}}_{ij} dt - \ddot{\hat{\rho}}_{ij})
\end{aligned}
\tag{36}
$$

Two theorems are obtained for the asymptotic identification of the unknown input and for the case when the unknown input is a smooth function.

Theorem 4. *Let Hypothesis 3 hold. Then the* ***exact*** *value of the unknown input ζ of the system (1) is estimated asymptotically by the algorithm (11), (12), (17), (19), (32), (33).*

Theorem 5. *Let Hypothesis 3 and 3.6 hold. Then the algorithm (11), (12), (17), (19), (32), (35). guarantees the identification of the unknown inputs in finite time.*

3.7 Fault Detection

Consider the case when the unknown inputs represent faults on the system. Consider the following system subject to faults in actuators and sensors:

$$
\begin{aligned}
\dot{x} &= Ax + Bu + E_a \zeta_a \\
y &= Cx + Du + F_s \zeta_s
\end{aligned}
\tag{37}
$$

where $x \in \mathbb{R}^n$ are the system state, $y \in \mathbb{R}^p$ are the system output, $u \in \mathbb{R}^{q_0}$ are the control inputs, $\zeta_a \in \mathbb{R}^{m_a}$ are the actuators faults, $\zeta_s \in \mathbb{R}^{m_s}$ are the sensors faults. Let $m = m_a + m_s$ and $m \leq p$.

Denote

$$\zeta = \begin{bmatrix} \zeta_a \\ \zeta_s \end{bmatrix}. \tag{38}$$

It is possible to rewrite the system in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + E\zeta \\ y &= Cx + Du + F\zeta \end{aligned}.$$

That is the general form (1) for unknown input observation, but here the matrixes E and F are defined as

$$E = [E_a \; 0_{p\times(m-m_a)}], \quad F = [0_{p\times(m-m_s)} \; F_s]. \tag{39}$$

The condition

$$rank \begin{bmatrix} E \\ F \end{bmatrix} = m$$

holds by definition.

Theorem 6. *Consider system (37) with faults, and let Hypothesis 3 hold. Algorithm (32) guarantees the finite time reconstruction of the vector faults in the form (38).*

Proof. The vector faults are reconstructed as unknown inputs. The proof of the identification of the unknown inputs was presented in the theorem 4.

Corollary 2. *If system (37) satisfies Hypothesis 3, then algorithm (32) guarantees the finite time reconstruction of discontinuous faults in sensors.*

Proof. The reconstruction of sensor faults is an algebraic operation with the known variables $\tilde{y}_3$, $\hat{e}_1$, $\hat{e}_2$. As the resulting reconstruction is algebraic, even discontinuous faults can be identified.

4 Example

The effectiveness of the observation and fault detection algorithm is tested on an example. Consider a linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + E\zeta \\ y &= Cx + F\zeta \end{aligned},$$

where $\zeta = [\zeta_a^T \; \zeta_s^T]^T$ is a faults vector with ζ_a being actuator faults, and ζ_s representing sensor faults. The values of the matrixes A, E, C, F are as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 3 & -1 & 2 \end{bmatrix}, \; E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are -1, 1, 1, 1, 1. Note that the system is unstable.

Let the actuators' fault be given by $\zeta_a(t) = 0.5\sin(2t) + 0.43$ appearing at $t = 7$. Let the sensor fault be a discontinuous signal that appear at $t = 10$.

The matrixes $F^{\perp}$ and $F^{\perp\perp}$ are obtained as

$$F^{\perp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; F^{\perp\perp} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

The system in the form (8), (9) takes on the form

$$\dot{\xi}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 3 & -1 & 2 \end{bmatrix} \xi_2 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \zeta$$

$$\begin{bmatrix} y_{2_1} \\ y_{2_2} \\ y_{3_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \xi_2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \zeta$$

The vector of partial relative degrees is (2, 2, 0).

Since the matrix A is unstable, the Luenberger gain L is chosen as

$$L = \begin{bmatrix} 7 & 19 & 31 & 0 & 0 \\ 0 & 0 & 0 & 11 & 41 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

The eigenvalues of the matrix $\tilde{A}_{22}$ are -1, -2, -3, -4, -5.

The matrix U_2 is given by

$$U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -11 & 1 \end{bmatrix}$$

The second order differentiator is applied to the components of the output y_2:

$$\begin{aligned} \dot{v}_{i_1} &= w_1 = -\alpha_3 N_i^{1/3} |v_1 - \tilde{y}_{2_i}|^{2/3} \operatorname{sign}(v_1 - \tilde{y}_{2_i}) + v_2, \\ \dot{v}_{i_2} &= w_2 = -\alpha_2 N_i^{1/2} |v_2 - w_1|^{1/2} \operatorname{sign}(v_2 - w_1) + v_3, \\ \dot{v}_{i_3} &= -\alpha_1 N_i \operatorname{sign}(v_3 - w_2). \end{aligned}$$

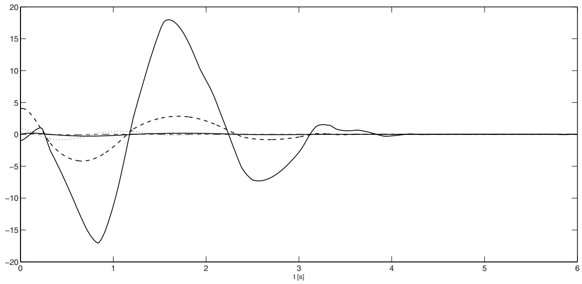

Fig. 1. Estimation error $x - \hat{x}$

Here N_i, $i = 1, 2$ take the values $N_1 = 1$, $N_2 = 8.5$ and the gains of the differentiator are $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$. The vector $\bar{v}$ is obtained as

$$\bar{v} = \begin{bmatrix} \bar{v}_{1_1} \\ \bar{v}_{1_2} \\ \bar{v}_{2_1} \\ \bar{v}_{2_2} \end{bmatrix}.$$

The matrix M_1 and the vector ρ_1 are given by

$$M_1 = \begin{bmatrix} 1 \; 18 \; 1 \; -1 \; -9 \end{bmatrix}, \tilde{\rho}_{1_1} = \begin{bmatrix} -2 \; 1 \; 0 \end{bmatrix} \begin{bmatrix} \bar{v}_{1_2} \\ \bar{v}_{2_2} \\ \int_0^t y_3 dt \end{bmatrix}$$

The first order differentiator (21) is applied to $\tilde{\rho}_{1_1}$:

$$\begin{aligned} \dot{\beta}_{1_1} &= \alpha_2 \bar{N}_1^{1/2} |\beta_{1_1} - \tilde{\rho}_{1_1}|^{1/2} sign(\beta_{1_1} - \tilde{\rho}_{1_1}) + \gamma_{1_1} \\ \dot{\gamma}_{1_1} &= \alpha_1 \bar{N}_1 sign(\gamma_{1_1} - \dot{\beta}_{1_1}) \end{aligned}$$

The matrix M_n and the vector ρ_n are given by

$$M_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -11 & 1 \\ 1 & 18 & 1 & -1 & -9 \end{bmatrix}, \; \rho_n = \begin{bmatrix} \bar{v}_{1_1} \\ \bar{v}_{1_2} \\ \bar{v}_{2_1} \\ \bar{v}_{2_2} \\ \gamma_{1_1} \end{bmatrix}$$

According to Theorem 3 the estimation error is presented in figure 1.

The estimation of the states x_1, x_5 is demonstrated in figure 2. The instability of the states are presented in figure 3.

Finally, the fault is reconstructed after the convergence of the observer, the fault reconstruction is shown in figure 4. Note that the sensor fault is discontinuous.

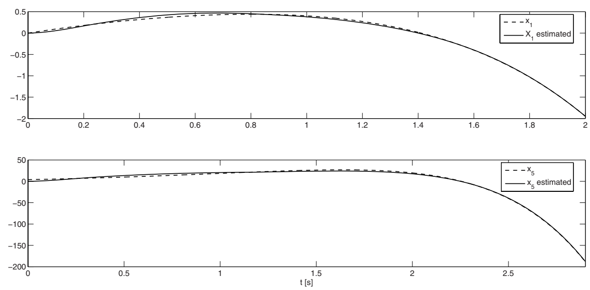

Fig. 2. Estimation of x_1 and x_5

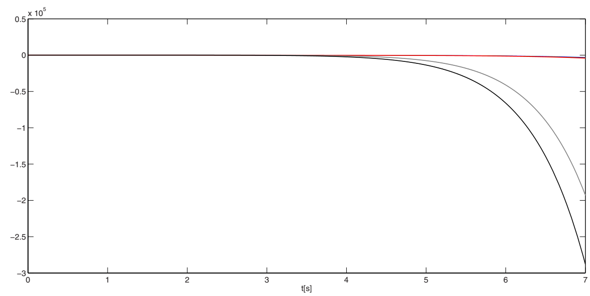

Fig. 3. System states

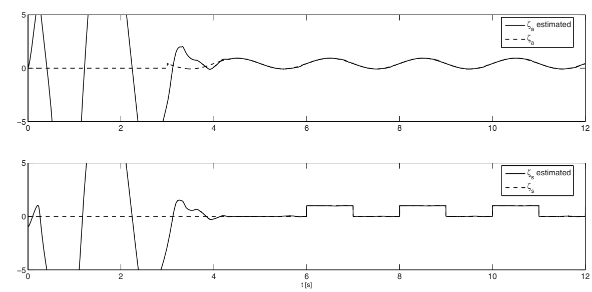

Fig. 4. Actuator fault reconstruction (above). Sensor fault reconstruction (below).

5 Possible Generalizations and Applications

5.1 Observer Design for Strongly Detectable Systems

In this case the weakly unobservable subspace $\mathcal{V}^*$ has non-zero dimension. The design of the observers for this case is considered in the works by [27], [34], [25].

5.2 Unknown Input Identification for Not Strongly Detectable Systems

The sufficient and necessary conditions for the identification of the unknown input, even for the case when the system is not strongly detectable are presented in [34] and [25].

5.3 Mechanical Systems

The main restriction for the generalization of the High-Order Sliding-Mode observer technique for the nonlinear systems is the necessity of the Bounded-Input Bounded-State (BIBS) properties. On the other hand the majority of mechanical systems satisfy the BIBS condition. It allows to design the second order sliding mode observers for mechanical systems. The equivalent output injection of the sliding mode technique is applied for perturbation and parameters' identification in the papers [13], [14].

5.4 Nonlinear Case

Local High-Order Sliding Mode observers for nonlinear systems with unknown inputs and with well defined vector relative degree were designed in [23].

5.5 Applications

HOSM observers are used in various applications.

In [35] second order sliding mode observers based on the modified super-twisting algorithm by [13] are applied for backlash identification.

A feedback linearization-based controller with a high order sliding mode parallel observer is applied in [36] to a quadrotor unmanned aerial vehicle. The model of the system has a vector relative degree (4, 4, 4, 2) with respect to the measurable outputs. A HOSM observer estimates the effects of the external disturbances, like a wind, for example.

In [37], [38], [39], [40], [41] HOSM observers are used for the estimation of vehicle and heavy cars parameters, such as stiffness, side sleep angles contact and vertical forces, tires longitudinal forces, road profile. Some of such applications are described in the chapter of this book by Shraim, Ouladsine, and Fridman.

The effectiveness of higher order sliding mode observers for fault detection was shown in [42], [25], [34]. The application of HOSM observers to the faults reconstruction in a leader/follower spacecraft system is considered in [43].

It is very important for control of bipeds to have the observers converging to exact values of legs and body velocities during the steps and finite time converging controllers. As it is shown in [44] (see the chapter of this book by Lebastard, Aoustin, Plestan, and Fridman). HOSM observers provide for the reasonable estimation of the bipeds variables.

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Appendix: Proofs

Proof of Theorem 2

Proof. Recall that U_{y_1} is the observability matrix for the pair (C_1, A). Following [29] the unobservable subspace of this pair is given by

$$\mathcal{N}_1 = \bigcap_{i=1}^{n} \ker(C_1 A^{i-1}) = ker U_{y_1}. \tag{40}$$

It is known ([29]) that $\mathcal{N}_1 \subset \mathcal{X}$ satisfies $A\mathcal{N}_1 \subset \mathcal{N}_1$, that is, the subspace $\mathcal{N}_1$ is A-invariant.

The inverse of the matrix T_y can be represented as

$$T_y^{-1} = [\, U_{y_1}^{+} \; \bar{U}_{y_1}^{+} \,],$$

where $U_{y_1}^{+} \in \mathbb{R}^{n \times n_{y_1}}$, $\bar{U}_{y_1}^{+} \in \mathbb{R}^{n \times (n-n_{y_1})}$.

Apply the transformation T_y to each matrix of (7):

$$T_y A T_y^{-1} = \begin{bmatrix} U_{y_1} A U_{y_1}^{+} & U_{y_1} A \bar{U}_{y_1}^{+} \\ \bar{U}_{y_1} A U_{y_1}^{+} & \bar{U}_{y_1} A \bar{U}_{y_1}^{+} \end{bmatrix}.$$

By definition $A\bar{U}_{y_1}^{+} \in \mathcal{N}_1$, and it is clear from equation (40) that

$$U_{y_1} A \bar{U}_{y_1}^{+} = 0_{n_{y_1} \times (n-n_{y_1})}.$$

The transformed matrix $T_y B$ consists of the matrixes $B_1 = U_{y_1} B$ and $B_2 = \bar{U}_{y_1} B$,

$$T_y B = \begin{bmatrix} U_{y_1} B \\ \bar{U}_{y_1} B \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

The transformed matrix E takes the form

$$T_y E = \begin{bmatrix} U_{y_1} \\ \bar{U}_{y_1} \end{bmatrix} E = \begin{bmatrix} U_{y_1} E \\ E_2 \end{bmatrix}.$$

It follows from definition 8 and the matrix (U_{y_1})being the observability matrix (U_{y_1}) of the pair (C_1, A) that $U_{y_1} E = 0$.

The transformed matrix C takes the form

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} T_y^{-1} = \begin{bmatrix} C_1 U_{y_1}^+ & C_1 \bar{U}_{y_1}^+ \\ C_2 U_{y_1}^+ & C_2 \bar{U}_{y_1}^+ \\ C_3 U_{y_1}^+ & C_3 \bar{U}_{y_1}^+ \end{bmatrix}.$$

Consider the matrixes C_1, C_2. It is clear from the definitions of $U_{y_1}^+$ and $\bar{U}_{y_1}^+$ that

$$C_1 \bar{U}_{y_1}^+ = 0_{p_1 \times (n - n_{y_1})}, \quad C_2 U_{y_1}^+ = 0_{p_2 \times n_{y_1}}.$$

The remaining submatrixes are given by

$$\begin{aligned} C_{11} &= C_1 U_{y_1}^+ \in \mathbb{R}^{p_1 \times n_{y_1}}, & C_{22} &= C_2 \bar{U}_{y_1}^+ \in \mathbb{R}^{p_2 \times (n - n_{y_1})}, \\ C_{31} &= C_3 U_{y_1}^+ \in \mathbb{R}^{p_3 \times n_{y_1}}, & C_{32} &= C_3 \bar{U}_{y_1}^+ \in \mathbb{R}^{p_3 \times (n - n_{y_1})}. \end{aligned}$$

The matrixes D_1, D_2, D_3 and F_3 have the same form as in (7). The theorem is proved.

Proof of Corollary 1

Proof. The rank of the observability matrix of the pair (C_1, A) is n_{y_1}, and is invariant under similarity transformations. Compute the observability matrix of $(C_1 T_y^{-1}, T_y A T_y^{-1})$ which has the form

$$\begin{bmatrix} C_1 T_y^{-1} \\ C_1 T_y^{-1}(T_y A T_y^{-1}) \\ C_1 T_y^{-1}(T_y A T_y^{-1})^2 \\ \vdots \\ C_1 T_y^{-1}(T_y A T_y^{-1})^{n-1} \end{bmatrix}.$$

It is known that $C_1 \bar{U}_{y_1}^+ = 0$ and $C_1 U_{y_1}^+ = C_{11}$, thus

$$\begin{bmatrix} C_1 T_y^{-1} \\ C_1 T_y^{-1}(T_y A T_y^{-1}) \\ C_1 T_y^{-1}(T_y A T_y^{-1})^2 \\ \vdots \\ C_1 T_y^{-1}(T_y A T_y^{-1})^{n-1} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{11} A_{11} & 0 \\ C_{11} A_{11}^2 & 0 \\ \vdots & \\ C_{11} A_{11}^{n-1} & 0 \end{bmatrix}.$$

Taking into account that the rank of the observability matrix of the pair (C_1, A) is n_{y_1}, obtain

$$\operatorname{rank}\begin{bmatrix} C_{11} & 0 \\ C_{11}A_{11} & 0 \\ C_{11}A_{11}^2 & 0 \\ \vdots & \\ C_{11}A_{11}^{n-1} & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} C_{11} \\ C_{11}A_{11} \\ C_{11}A_{11}^2 \\ \vdots \\ C_{11}A_{11}^{n-1} \end{bmatrix} = n_{y_1}.$$

Note that by definition the rank of the last matrix is equal to the rank of the reduced order matrix, therefore

$$\operatorname{rank}\begin{bmatrix} C_{11} \\ C_{11}A_{11} \\ C_{11}A_{11}^2 \\ \vdots \\ C_{11}A_{11}^{n_{y_1}-1} \end{bmatrix} = n_{y_1}.$$

The last matrix is just the observability matrix of the pair (C_{11}, A_{11}) corresponding to the reduced order system (10). Hence, the observability matrix has the rank n_{y_1}, and the subsystem (10) is observable.

Proof of Lemma 2

Proof. Denoting $\tilde{\sigma}_{ij} = \sigma_{ij}/N_i$ obtain that

$$\begin{aligned}
&\dot{\tilde{\sigma}}_{i1} = -\alpha_{n_{1i}}|\tilde{\sigma}_{i1}|^{(n_{1i}-1)/n_{1i}}\operatorname{sign}(\sigma_{i1}) + \tilde{\sigma}_{i2},\\
&\dot{\tilde{\sigma}}_{i2} = -\alpha_{n_{1i}-1}|\tilde{\sigma}_{i2} - \dot{\tilde{\sigma}}_{i1}|^{(n_{1i}-2)/(n_{1i}-1)}\operatorname{sign}(\sigma_{i2} - \dot{\tilde{\sigma}}_{i1}) + \tilde{\sigma}_{i3},\\
&\vdots\\
&\dot{\tilde{\sigma}}_{i,n_{1i}-1} = -\alpha_2|\tilde{\sigma}_{i,n_{1i}-1} - \dot{\tilde{\sigma}}_{i,n_{1i}-2}|^{1/2}\operatorname{sign}(\tilde{\sigma}_{i,n_{1i}-1} - \dot{\tilde{\sigma}}_{i,n_{1i}-2}) + \tilde{\sigma}_{i,n_{1i}},\\
&\dot{\tilde{\sigma}}_{i,n_{1i}} \in -\alpha_1\operatorname{sign}(\tilde{\sigma}_{i,n_{1i}} - \dot{\tilde{\sigma}}_{i,n_{1i}-1}) + [-1, 1].
\end{aligned}$$

The Lemma is now a direct consequence of Lemma 8 from ([22]).